

PRODUCTS OF COMMUTATORS IN A LIE NILPOTENT ASSOCIATIVE ALGEBRA

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ABSTRACT. Let F be a field and let $F\langle X \rangle$ be the free unital associative algebra over F freely generated by an infinite countable set $X = \{x_1, x_2, \dots\}$. Define a left-normed commutator $[a_1, a_2, \dots, a_n]$ recursively by $[a_1, a_2] = a_1a_2 - a_2a_1$, $[a_1, \dots, a_{n-1}, a_n] = [[a_1, \dots, a_{n-1}], a_n]$ ($n \geq 3$). For $n \geq 2$, let $T^{(n)}$ be the two-sided ideal in $F\langle X \rangle$ generated by all commutators $[a_1, a_2, \dots, a_n]$ ($a_i \in F\langle X \rangle$).

Let F be a field of characteristic 0. In 2008 Etingof, Kim and Ma conjectured that $T^{(m)}T^{(n)} \subset T^{(m+n-1)}$ if and only if m or n is odd. In 2010 Bapat and Jordan confirmed the “if” direction of the conjecture: if at least one of the numbers m, n is odd then $T^{(m)}T^{(n)} \subset T^{(m+n-1)}$. The aim of the present note is to confirm the “only if” direction of the conjecture. We prove that if $m = 2m'$ and $n = 2n'$ are even then $T^{(m)}T^{(n)} \not\subset T^{(m+n-1)}$. Our result is valid over any field F .

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1. INTRODUCTION

Let F be a field. Let $X = \{x_1, x_2, \dots\}$ be an infinite countable set and let $F\langle X \rangle$ be the free associative algebra over F freely generated by X . Define a left-normed commutator $[a_1, a_2, \dots, a_n]$ recursively by $[a_1, a_2] = a_1a_2 - a_2a_1$, $[a_1, \dots, a_{n-1}, a_n] = [[a_1, \dots, a_{n-1}], a_n]$ ($n \geq 3$). For $n \geq 2$, let $T^{(n)}$ be the two-sided ideal in $F\langle X \rangle$ generated by all commutators $[a_1, a_2, \dots, a_n]$ ($a_i \in F\langle X \rangle$).

In 2008 Etingof, Kim and Ma [9] made a conjecture (see Conjecture 3.6 in the arXiv version of [9]) that can be reformulated as follows:

Conjecture 1.1 (see [9]). *Let F be a field of characteristic 0. Then $T^{(m)}T^{(n)} \subset T^{(m+n-1)}$ if and only if m or n is odd.*

In [9] this conjecture was confirmed for m and n such that $m+n \leq 7$. In 2010 Bapat and Jordan [2, Corollary 1.4] confirmed the “if” direction of the conjecture for arbitrary m, n .

Theorem 1.2 (see [2]). *Let F be a field of characteristic $\neq 2, 3$. Let $m, n \in \mathbb{Z}$, $m, n > 1$ and at least one of the numbers m, n is odd. Then*

$$(1) \quad T^{(m)}T^{(n)} \subset T^{(m+n-1)}.$$

The aim of the present note is to confirm the “only if” direction of the conjecture. Our main result is as follows.

Theorem 1.3. *Let F be a field and let $m = 2m'$, $n = 2n'$ be arbitrary positive even integers. Then*

$$T^{(m)}T^{(n)} \not\subset T^{(m+n-1)}.$$

Recall that an associative algebra A is Lie nilpotent of class at most c if $[u_1, \dots, u_c, u_{c+1}] = 0$ for all $u_i \in A$. We deduce Theorem 1.3 from the following result.

Theorem 1.4. *Let F be a field and let $m = 2m'$, $n = 2n'$ be arbitrary positive even integers. Then there exists a unital associative algebra A such that the following two conditions are satisfied:*

i) for all $u_1, u_2, \dots, u_{m+n-1} \in A$ we have

$$[u_1, u_2, \dots, u_{m+n-1}] = 0,$$

that is, the algebra A is Lie nilpotent of class at most $m+n-2$;

ii) there are $v_1, \dots, v_m, w_1, \dots, w_n \in A$ such that

$$[v_1, \dots, v_m][w_1, \dots, w_n] \neq 0.$$

If F is a field of characteristic $\neq 2$ then in Theorem 1.4 one can take $A = E \otimes E_r$ where E is the infinite-dimensional unital Grassmann algebra and E_r is the r -generated unital Grassmann algebra for $r = m + n - 4$.

Remarks. 1. Note that if $k > \ell$ then $T^{(k)} \subset T^{(\ell)}$; in particular, $T^{(m+n-1)} \subset T^{(m+n-2)}$. Let R be an arbitrary associative and commutative unital ring and let $m, n \in \mathbb{Z}$, $m, n > 1$. Then in $R\langle X \rangle$ we have

$$T^{(m)}T^{(n)} \subset T^{(m+n-2)}.$$

This assertion was proved by Latyshev [15, Lemma 1] in 1965 (Latyshev's paper was published in Russian) and independently rediscovered by Gupta and Levin [13, Theorem 3.2] in 1983.

2. The proof of Theorem 1.2 given in [2] is valid for algebras over an associative and commutative unital ring R such that $\frac{1}{6} \in R$. In fact, Theorem 1.2 holds over any R such that $\frac{1}{3} \in R$ (see [1, Remark 3.9] for explanation). Moreover, for some m and n (1) holds over an arbitrary ring R : for instance, $T^{(3)}T^{(3)} \subset T^{(5)}$ in $R\langle X \rangle$ for any R (see [5, Lemma 2.1]). However, in general Theorem 1.2 fails over \mathbb{Z} and over a field of characteristic 3: it was shown in [7, 14] that in this case $T^{(3)}T^{(2)} \not\subset T^{(4)}$ and moreover, $T^{(3)}(T^{(2)})^\ell \not\subset T^{(4)}$ for all $\ell \geq 1$.

3. In 1978 Volichenko proved Theorem 1.2 for $m = 3$ and arbitrary n in the preprint [16] written in Russian; in 2007 Gordienko [10] independently proved this theorem for $m = 3, n = 2$. These results were unknown to the authors of [2, 9]. Recently another proof of Theorem 1.2 has been published in [11].

4. In [9] a pair (m, n) of positive integers was called *null* if for each algebra A (over a field F of characteristic 0) $T^{(m)}(A)T^{(n)}(A) \subset T^{(m+n-1)}(A)$ where $T^{(\ell)}(A)$ is the two-sided ideal in A generated by all commutators $[a_1, \dots, a_\ell]$ ($a_i \in A$). The original conjecture stated in [9, Conjecture 3.6] was as follows: A pair (m, n) is null if and only if m or n is odd. This conjecture is equivalent to Conjecture 1.1 above; this can be checked using the same argument that is used to deduce Theorem 1.3 from Theorem 1.4.

2. PROOFS OF THEOREMS 1.3 AND 1.4

First we prove some auxiliary results.

Let G and H be unital associative algebras over a field F such that $[g_1, g_2, g_3] = 0$, $[h_1, h_2, h_3] = 0$ for all $g_i \in G$, $h_j \in H$. Note that each commutator $[g_1, g_2]$ ($g_i \in G$) is central in G , that is, $[g_1, g_2]g = g[g_1, g_2]$ for each $g \in G$. Similarly, each commutator $[h_1, h_2]$ ($h_j \in H$) is central in H .

Lemma 2.1. *Let*

$$c_\ell = [g_1 \otimes h_1, g_2 \otimes h_2, \dots, g_\ell \otimes h_\ell]$$

where $\ell \geq 2$, $g_i \in G, h_j \in H$. Then

$$\begin{aligned} c_2 &= [g_1, g_2] \otimes h_1 h_2 + g_2 g_1 \otimes [h_1, h_2], \\ c_{2k} &= [g_1, g_2][g_3, g_4] \dots [g_{2k-1}, g_{2k}] \otimes [h_1 h_2, h_3][h_4, h_5] \dots [h_{2k-2}, h_{2k-1}] h_{2k} \\ &\quad + [g_2 g_1, g_3][g_4, g_5] \dots [g_{2k-2}, g_{2k-1}] g_{2k} \otimes [h_1, h_2][h_3, h_4] \dots [h_{2k-1}, h_{2k}] \quad (k > 1), \\ c_{2k+1} &= [g_1, g_2][g_3, g_4] \dots [g_{2k-1}, g_{2k}] g_{2k+1} \otimes [h_1 h_2, h_3][h_4, h_5] \dots [h_{2k}, h_{2k+1}] \\ &\quad + [g_2 g_1, g_3][g_4, g_5] \dots [g_{2k}, g_{2k+1}] \otimes [h_1, h_2][h_3, h_4] \dots [h_{2k-1}, h_{2k}] h_{2k+1} \quad (k \geq 1). \end{aligned}$$

Proof. Induction on the length ℓ of the commutator c_ℓ . If $\ell = 2$ then

$$\begin{aligned} c_2 &= [g_1 \otimes h_1, g_2 \otimes h_2] = g_1 g_2 \otimes h_1 h_2 - g_2 g_1 \otimes h_2 h_1 \\ &= g_1 g_2 \otimes h_1 h_2 - g_2 g_1 \otimes h_1 h_2 + g_2 g_1 \otimes h_1 h_2 - g_2 g_1 \otimes h_2 h_1 \\ &= [g_1, g_2] \otimes h_1 h_2 + g_2 g_1 \otimes [h_1, h_2]. \end{aligned}$$

Let $\ell > 2$; suppose that for each $\ell' < \ell$ the lemma has already been proved.

Let $\ell = 2k + 1$ ($k \geq 1$). By the induction hypothesis, we have

$$\begin{aligned} c_{2k+1} &= [c_{2k}, g_{2k+1} \otimes h_{2k+1}] \\ &= [[g_1, g_2] \cdots [g_{2k-1}, g_{2k}] \otimes [h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k-2}, h_{2k-1}] h_{2k}, g_{2k+1} \otimes h_{2k+1}] \\ &\quad + [[g_2 g_1, g_3][g_4, g_5] \cdots [g_{2k-2}, g_{2k-1}] g_{2k} \otimes [h_1, h_2] \cdots [h_{2k-1}, h_{2k}], g_{2k+1} \otimes h_{2k+1}]. \end{aligned}$$

Note that the products $[g_1, g_2] \cdots [g_{2k-1}, g_{2k}]$ and $[h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k-2}, h_{2k-1}]$ are central in G and H , respectively, so

$$\begin{aligned} & [[g_1, g_2] \cdots [g_{2k-1}, g_{2k}] \otimes [h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k-2}, h_{2k-1}] h_{2k}, g_{2k+1} \otimes h_{2k+1}] \\ &= [g_1, g_2] \cdots [g_{2k-1}, g_{2k}] g_{2k+1} \otimes [h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k-2}, h_{2k-1}] h_{2k} h_{2k+1} \\ &\quad - g_{2k+1} [g_1, g_2] \cdots [g_{2k-1}, g_{2k}] \otimes h_{2k+1} [h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k-2}, h_{2k-1}] h_{2k} \\ &= [g_1, g_2] \cdots [g_{2k-1}, g_{2k}] g_{2k+1} \otimes [h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k-2}, h_{2k-1}] h_{2k} h_{2k+1} \\ &\quad - [g_1, g_2] \cdots [g_{2k-1}, g_{2k}] g_{2k+1} \otimes [h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k-2}, h_{2k-1}] h_{2k+1} h_{2k} \\ &= [g_1, g_2] \cdots [g_{2k-1}, g_{2k}] g_{2k+1} \otimes [h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k-2}, h_{2k-1}] [h_{2k}, h_{2k+1}]. \end{aligned}$$

Similarly,

$$\begin{aligned} & [[g_2 g_1, g_3][g_4, g_5] \cdots [g_{2k-2}, g_{2k-1}] g_{2k} \otimes [h_1, h_2] \cdots [h_{2k-1}, h_{2k}], g_{2k+1} \otimes h_{2k+1}] \\ &= [g_2 g_1, g_3][g_4, g_5] \cdots [g_{2k-2}, g_{2k-1}] [g_{2k}, g_{2k+1}] \otimes [h_1, h_2] \cdots [h_{2k-1}, h_{2k}] h_{2k+1} \end{aligned}$$

so

$$\begin{aligned} c_{2k+1} &= [g_1, g_2] \cdots [g_{2k-1}, g_{2k}] g_{2k+1} \otimes [h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k}, h_{2k+1}] \\ &\quad + [g_2 g_1, g_3][g_4, g_5] \cdots [g_{2k}, g_{2k+1}] \otimes [h_1, h_2] \cdots [h_{2k-1}, h_{2k}] h_{2k+1}, \end{aligned}$$

as required.

Let $\ell = 2k$ ($k > 1$). By the induction hypothesis, we have

$$\begin{aligned} c_{2k} &= [c_{2k-1}, g_{2k} \otimes h_{2k}] \\ &= [[g_1, g_2] \cdots [g_{2k-3}, g_{2k-2}] g_{2k-1} \otimes [h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k-2}, h_{2k-1}], g_{2k} \otimes h_{2k}] \\ &\quad + [[g_2 g_1, g_3][g_4, g_5] \cdots [g_{2k-2}, g_{2k-1}] \otimes [h_1, h_2] \cdots [h_{2k-3}, h_{2k-2}] h_{2k-1}, g_{2k} \otimes h_{2k}] \\ &= [g_1, g_2] \cdots [g_{2k-3}, g_{2k-2}] [g_{2k-1}, g_{2k}] \otimes [h_1 h_2, h_3][h_4, h_5] \cdots [h_{2k-2}, h_{2k-1}] h_{2k} \\ &\quad + [g_2 g_1, g_3][g_4, g_5] \cdots [g_{2k-2}, g_{2k-1}] g_{2k} \otimes [h_1, h_2] \cdots [h_{2k-3}, h_{2k-2}] [h_{2k-1}, h_{2k}], \end{aligned}$$

as required.

This completes the proof of Lemma 2.1. □

Corollary 2.2. *Suppose that*

$$(2) \quad [f_1, f_2] \cdots [f_{2k-1}, f_{2k}] = 0 \quad \text{for all } f_j \in H.$$

Then for all $u_i \in G \otimes H$ we have

$$[u_1, u_2, \dots, u_{2k+1}] = 0.$$

Proof. Since each $u_i \in G \otimes H$ is a sum of products of the form $g \otimes h$ ($g \in G, h \in H$), the commutator $[u_1, u_2, \dots, u_{2k+1}]$ is a sum of commutators of the form $[g_1 \otimes h_1, g_2 \otimes h_2, \dots, g_{2k+1} \otimes h_{2k+1}]$. On the other hand, it follows from (2) and Lemma 2.1 that $[g_1 \otimes h_1, g_2 \otimes h_2, \dots, g_{2k+1} \otimes h_{2k+1}] = 0$ for all $g_i \in G, h_j \in H$. Thus, $[u_1, u_2, \dots, u_{2k+1}] = 0$ for all $u_i \in G \otimes H$, as required. □

Corollary 2.3. *Let $v_1 = g_1 \otimes 1$, $v_i = g_i \otimes h_i$ ($i = 2, \dots, 2m' - 1$), $v_{2m'} = g_{2m'} \otimes 1$, $w_1 = g'_1 \otimes 1$, $w_j = g'_j \otimes h'_j$ ($j = 2, \dots, 2n' - 1$), $w_{2n'} = g'_{2n'} \otimes 1$ where $g_i, g'_i \in G, h_j, h'_j \in H$. Then*

$$\begin{aligned} [v_1, \dots, v_{2m'}][w_1, \dots, w_{2n'}] &= [g_1, g_2] \cdots [g_{2m'-1}, g_{2m'}][g'_1, g'_2] \cdots [g'_{2n'-1}, g'_{2n'}] \\ &\quad \otimes [h_2, h_3] \cdots [h_{2m'-2}, h_{2m'-1}][h'_2, h'_3] \cdots [h'_{2n'-2}, h'_{2n'-1}]. \end{aligned}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned} [v_1, \dots, v_{2m'}] &= [g_1, g_2] \dots [g_{2m'-1}, g_{2m'}] \otimes [h_2, h_3] \dots [h_{2m'-2}, h_{2m'-1}], \\ [w_1, \dots, w_{2n'}] &= [g'_1, g'_2] \dots [g'_{2n'-1}, g'_{2n'}] \otimes [h'_2, h'_3] \dots [h'_{2n'-2}, h'_{2n'-1}]. \end{aligned}$$

The result follows. \square

Proof of Theorem 1.4. Two cases are to be considered: the case when $\text{char } F \neq 2$ and the case when $\text{char } F = 2$.

Case 1. Suppose that F is a field of characteristic $\neq 2$. Let E be the unital infinite-dimensional Grassmann (or exterior) algebra over F . Then E is generated by the elements e_i ($i = 1, 2, \dots$) such that $e_i e_j = -e_j e_i$, $e_i^2 = 0$ for all i, j and the set

$$\mathcal{B} = \{e_{i_1} e_{i_2} \dots e_{i_k} \mid k \geq 0, i_1 < i_2 < \dots < i_k\}$$

forms a basis of E over F .

It is well known that $[g_1, g_2, g_3] = 0$ for all $g_i \in E$. Indeed, we may assume without loss of generality that $g_\ell \in \mathcal{B}$ ($\ell = 1, 2, 3$). Let $g_\ell = e_{i_{\ell 1}} \dots e_{i_{\ell k(\ell)}}$ ($\ell = 1, 2, 3$). Note that if $k = 2k'$ is even then the product $e_{i_1} e_{i_2} \dots e_{i_k}$ is central in E because it commutes with all generators e_i . Hence, if $k(1)$ or $k(2)$ is even then $[g_1, g_2] = 0$ and, therefore, $[g_1, g_2, g_3] = 0$. On the other hand, if both $k(1)$ and $k(2)$ are odd then the commutator $[g_1, g_2] = 2g_1 g_2 = 2e_{i_{11}} \dots e_{i_{1k(1)}} e_{i_{21}} \dots e_{i_{2k(2)}}$ is central in E and again $[g_1, g_2, g_3] = 0$, as claimed.

Recall that the r -generated unital Grassmann algebra E_r is the unital subalgebra of E generated by e_1, e_2, \dots, e_r . Note that $[h_1, h_2, h_3] = 0$ for all $h_j \in E_r$.

Take $A = E \otimes E_r$ where $r = m + n - 4 = 2(m' + n' - 2)$. We can apply Lemma 2.1 and Corollaries 2.2 and 2.3 for $G = E$, $H = E_r$.

Let $k = m' + n' - 1$. Note that $2k > r$. It follows that $[f_1, f_2] \dots [f_{2k-1}, f_{2k}] = 0$ for all $f_i \in E_r$. Indeed, for all $f, f' \in E_r$ the commutator $[f, f']$ belongs to the linear span of the set $\{e_{i_1} \dots e_{i_{2\ell}} \mid \ell \geq 1, 1 \leq i_s \leq r\}$. Hence, $[f_1, f_2] \dots [f_{2k-1}, f_{2k}]$ belongs to the linear span of the set $\{e_{i_1} \dots e_{i_{2\ell}} \mid \ell \geq k, 1 \leq i_s \leq r\}$. Since $2\ell \geq 2k > r$, each product $e_{i_1} \dots e_{i_{2\ell}}$ above contains equal terms $e_{i_s} = e_{i_{s'}}$ ($s < s'$) and, therefore, is equal to 0. Thus, $[f_1, f_2] \dots [f_{2k-1}, f_{2k}] = 0$, as claimed.

Now, by Corollary 2.2, we have $[u_1, \dots, u_{2k+1}] = 0$ for all $u_i \in E \otimes E_r$, that is,

$$[u_1, \dots, u_{m+n-1}] = 0$$

for all $u_1, \dots, u_{m+n-1} \in A$, as required.

Further, take $v_1 = e_1 \otimes 1$, $v_i = e_i \otimes e_{i-1}$ ($i = 2, \dots, 2m' - 1$), $v_{2m'} = e_{2m'} \otimes 1$, $w_1 = e_{2m'+1} \otimes 1$, $w_j = e_{2m'+j} \otimes e_{2m'+j-3}$ ($j = 2, \dots, 2n' - 1$), $w_{2n'} = e_{2m'+2n'} \otimes 1$. Note that if $i \neq j$ then $[e_i, e_j] = 2e_i e_j$. By Corollary 2.3, we have

$$\begin{aligned} [v_1, \dots, v_{2m'}][w_1, \dots, w_{2n'}] &= [e_1, e_2] \dots [e_{2m'-1}, e_{2m'}][e_{2m'+1}, e_{2m'+2}] \dots [e_{2m'+2n'-1}, e_{2m'+2n'}] \\ &\quad \otimes [e_1, e_2] \dots [e_{2m'-3}, e_{2m'-2}][e_{2m'-1}, e_{2m'}] \dots [e_{2m'+2n'-5}, e_{2m'+2n'-4}] \\ &= 2^{m'+n'} e_1 e_2 \dots e_{2m'-1} e_{2m'} e_{2m'+1} e_{2m'+2} \dots e_{2m'+2n'-1} e_{2m'+2n'} \\ &\quad \otimes 2^{m'+n'-2} e_1 e_2 \dots e_{2m'-3} e_{2m'-2} e_{2m'-1} e_{2m'} \dots e_{2m'+2n'-5} e_{2m'+2n'-4} \\ &= 2^{m+n-2} e_1 e_2 \dots e_{m+n} \otimes e_1 e_2 \dots e_{m+n-4} \neq 0, \end{aligned}$$

as required.

Case 2. Suppose that F is a field of characteristic 2. Let \mathcal{G} be the group given by the presentation

$$\mathcal{G} = \langle y_1, y_2, \dots \mid y_i^2 = 1, ((y_i, y_j), y_k) = 1 \ (i, j, k = 1, 2, \dots) \rangle$$

where $(a, b) = a^{-1} b^{-1} a b$. Then \mathcal{G} is a nilpotent group of class 2 so $(a, b)c = c(a, b)$ and $(a, bc) = (a, b)(a, c)$ for all $a, b, c \in \mathcal{G}$. The quotient group \mathcal{G}/\mathcal{G}' is an elementary abelian 2-group so $a^2 \in \mathcal{G}'$ for all $a \in \mathcal{G}$. Hence, $(a, b)^2 = (a^2, b) = 1$ and $(a, b) = (a, b)^{-1} = (b, a)$ for all $a, b \in \mathcal{G}$.

Let $(<)$ be an arbitrary linear order on the set $\{(i, j) \mid i, j \in \mathbb{Z}, 0 < i < j\}$. The following lemma is well known and easy to check.

Lemma 2.4. *Let $a \in \mathcal{G}$. Then a can be written in a unique way in the form*

$$(3) \quad a = y_{i_1} \cdots y_{i_q}(y_{j_1}, y_{j_2}) \cdots (y_{j_{2q'-1}}, y_{j_{2q'}}) \\ \text{where } q, q' \geq 0; \quad i_1 < \cdots < i_q, \quad j_{2s-1} < j_{2s} \text{ for all } s, \quad (j_{2s-1}, j_{2s}) < (j_{2s'-1}, j_{2s'}) \text{ if } s < s'.$$

Let $F\mathcal{G}$ be the group algebra of \mathcal{G} over F . Let $d_{ij} = (y_i, y_j) + 1 \in F\mathcal{G}$. Note that $d_{ij} = d_{ji}$ and $d_{ii} = 0$ for all i, j .

Let I be the two-sided ideal of $F\mathcal{G}$ generated by the set

$$S = \{d_{i_1 i_2} d_{i_3 i_4} + d_{i_1 i_3} d_{i_2 i_4} \mid i_1, i_2, i_3, i_4 = 1, 2, \dots\}.$$

Note that $d_{j_1 j_3} d_{j_2 j_3} \in I$ for all j_1, j_2, j_3 because $d_{j_1 j_3} d_{j_2 j_3} = d_{j_1 j_3} d_{j_2 j_3} + d_{j_1 j_2} d_{j_3 j_3} \in S$. Since $d_{ij} = d_{ji}$ for all i, j , we have $d_{i_1 i_2} d_{i_3 i_4} \in I$ if any two of the indices i_1, i_2, i_3, i_4 coincide. It follows that

$$(4) \quad \prod_s (y_j, y_{i_s}) + 1 = \prod_s (d_{ji_s} + 1) + 1 = \left(\prod_s d_{ji_s} + \cdots + \sum_{s < s'} d_{ji_s} d_{ji_{s'}} + \sum_s d_{ji_s} + 1 \right) + 1 \equiv \sum_s d_{ji_s} \pmod{I}.$$

The following two lemmas are well known (see, for instance, [12, Lemma 2.1], [13, Example 3.8]).

Lemma 2.5. *For all $u_1, u_2, u_3 \in F\mathcal{G}$, we have $[u_1, u_2, u_3] \in I$.*

Proof. Let $c = \prod_s y_{i_s} \in \mathcal{G}$. Using (4), we have

$$\begin{aligned} ((y_{j_1}, c) + 1)((y_{j_2}, c) + 1) &= \left(\prod_s (y_{j_1}, y_{i_s}) + 1 \right) \left(\prod_s (y_{j_2}, y_{i_s}) + 1 \right) \\ &\equiv \left(\sum_s d_{j_1 i_s} \right) \left(\sum_s d_{j_2 i_s} \right) \pmod{I} = \sum_s d_{j_1 i_s} d_{j_2 i_s} + \sum_{s < s'} (d_{j_1 i_s} d_{j_2 i_{s'}} + d_{j_1 i_{s'}} d_{j_2 i_s}) \equiv 0 \pmod{I}, \end{aligned}$$

that is, $((y_{j_1}, c) + 1)((y_{j_2}, c) + 1) \in I$ for all $c \in \mathcal{G}$ and all j_1, j_2 . Similar to (4), one can check that

$$(5) \quad \prod_s (y_{i_s}, c) + 1 \equiv \sum_s ((y_{i_s}, c) + 1) \pmod{I}.$$

Let $a, b \in \mathcal{G}$, $a = \prod_s y_{i_s}$, $b = \prod_{s'} y_{i'_{s'}}$. Using (5), we have

$$\begin{aligned} ((a, c) + 1)((b, c) + 1) &= \left(\prod_s (y_{i_s}, c) + 1 \right) \left(\prod_{s'} (y_{i'_{s'}}, c) + 1 \right) \\ &\equiv \left(\sum_s ((y_{i_s}, c) + 1) \right) \left(\sum_{s'} ((y_{i'_{s'}}, c) + 1) \right) \pmod{I} = \sum_{s, s'} ((y_{i_s}, c) + 1)((y_{i'_{s'}}, c) + 1) \equiv 0 \pmod{I}, \end{aligned}$$

that is,

$$(6) \quad ((a, c) + 1)((b, c) + 1) \in I \quad \text{for all } a, b, c \in \mathcal{G}.$$

Now we are in a position to complete the proof of Lemma 2.5. It is clear that it suffices to prove that $[a, b, c] \in I$ for all $a, b, c \in \mathcal{G}$. Note that, for $a, b \in \mathcal{G}$,

$$[a, b] = ab(1 + b^{-1}a^{-1}ba) = ab(1 + (b, a))$$

(recall that $\text{char } F = 2$). We have

$$[a, b, c] = [ab(1 + (b, a)), c] = [ab, c](1 + (b, a)) = abc(1 + (c, ab))(1 + (b, a)) = abc(1 + (c, ab))(1 + (b, ab))$$

because $(b, ab) = (b, a)(b, b) = (b, a)$. By (6), we have $(1 + (c, ab))(1 + (b, ab)) \in I$ and therefore $[a, b, c] \in I$, as required. \square

Lemma 2.6. *For all $\ell > 0$, we have $((y_1, y_2) + 1) \cdots ((y_{2\ell-1}, y_{2\ell}) + 1) \notin I$.*

Proof. Let \mathcal{G}' be the derived subgroup of \mathcal{G} ; let $c_{ij} = (y_i, y_j)$. Then each element of \mathcal{G}' can be written in a unique way in the form $c_{j_1 j_2} \dots c_{j_{2q-1} j_{2q}}$ where $q \geq 0$, $j_{2s-1} < j_{2s}$ for all s , $(j_{2s-1}, j_{2s}) < (j_{2s'-1}, j_{2s'})$ if $s < s'$.

Let $F\mathcal{G}'$ be the group algebra of \mathcal{G}' over F , $F\mathcal{G}' \subset F\mathcal{G}$. Recall that $d_{ij} = c_{ij} + 1$. Since the set

$$\mathcal{G}' = \{c_{j_1 j_2} \dots c_{j_{2q-1} j_{2q}} \mid q \geq 0; j_{2s-1} < j_{2s} \text{ for all } s; (j_{2s-1}, j_{2s}) < (j_{2s'-1}, j_{2s'}) \text{ if } s < s'\}$$

is a basis of $F\mathcal{G}'$ over F , so is the set

$$\{d_{j_1 j_2} \dots d_{j_{2q-1} j_{2q}} \mid q \geq 0; j_{2s-1} < j_{2s} \text{ for all } s; (j_{2s-1}, j_{2s}) < (j_{2s'-1}, j_{2s'}) \text{ if } s < s'\}.$$

It follows that $F\mathcal{G}'$ is a unital F -algebra generated by pairwise commuting elements d_{ij} subject to the relations $d_{ij}^2 = 0$, $d_{ij} = d_{ji}$ for all i, j and $d_{ii} = 0$ for all i .

By Lemma 2.4, the group \mathcal{G} is a disjoint union of the sets $y_{i_1} \dots y_{i_q} \mathcal{G}'$ ($q \geq 0, 0 < i_1 < i_2 < \dots < i_q$). Hence, $F\mathcal{G}$ is a direct sum of the vector subspaces $y_{i_1} \dots y_{i_q} F\mathcal{G}'$,

$$F\mathcal{G} = \bigoplus_{q \geq 0, 0 < i_1 < i_2 < \dots < i_q} y_{i_1} \dots y_{i_q} F\mathcal{G}'.$$

Recall that I is a two-side ideal of $F\mathcal{G}$ generated by S . Since S is central in $F\mathcal{G}$, we have

$$I = F\mathcal{G} \cdot S = \bigoplus_{q \geq 0, 0 < i_1 < i_2 < \dots < i_q} y_{i_1} \dots y_{i_q} F\mathcal{G}' \cdot S.$$

It follows that $I \cap F\mathcal{G}' = F\mathcal{G}' \cdot S$ so to prove the lemma one has to check that $d_{12} \dots d_{(2\ell-1)2\ell} \notin F\mathcal{G}' \cdot S$, that is, to check that the product $d_{12} \dots d_{(2\ell-1)2\ell}$ does not belong to the ideal of $F\mathcal{G}'$ generated by S . However, this is the case because the set S consists of the elements $d_{i_1 i_2} d_{i_3 i_4} + d_{i_1 i_3} d_{i_2 i_4}$.

Indeed, let $P = F[t_i \mid i = 1, 2, \dots]$ be the F -algebra of (commutative) polynomials in t_i and let \mathcal{I} be the ideal of P generated by the set $\{t_i^2 \mid i = 1, 2, \dots\}$. Then the map $\psi(d_{ij}) \rightarrow t_i t_j + \mathcal{I}$ can be extended up to a homomorphism $F\mathcal{G}' \rightarrow P/\mathcal{I}$ because $\psi(d_{ij}^2) \equiv 0 \pmod{\mathcal{I}}$, $\psi(d_{ij}) = \psi(d_{ji})$ and $\psi(d_{ii}) \equiv 0 \pmod{\mathcal{I}}$. Since $\psi(d_{i_1 i_2} d_{i_3 i_4} + d_{i_1 i_3} d_{i_2 i_4}) = 2t_{i_1} t_{i_2} t_{i_3} t_{i_4} + \mathcal{I} = \mathcal{I}$ (recall that $\text{char } F = 2$), we have $\psi(S) = 0$. However, $\psi(d_{12} \dots d_{(2\ell-1)2\ell}) = t_1 \dots t_{2\ell} + \mathcal{I} \neq 0$ so $d_{12} \dots d_{(2\ell-1)2\ell} \notin F\mathcal{G}' \cdot S = I \cap F\mathcal{G}'$ and, therefore, $d_{12} \dots d_{(2\ell-1)2\ell} \notin I$, as required. \square

Now we are in a position to complete the proof of Theorem 1.4. Let \mathcal{G}_r be the subgroup of \mathcal{G} generated by y_1, \dots, y_r ; let $I_r = I \cap F\mathcal{G}_r$. Take $G = F\mathcal{G}/I$, $H = F\mathcal{G}_r/I_r$ where $r = m + n - 4 = 2(m' + n' - 2)$. Take $A = G \otimes H$. By Lemma 2.5, we can apply Lemma 2.1 and Corollaries 2.2 and 2.3.

Let $k = m' + n' - 1$; note that $2k > r$. We claim that $[f_1, f_2] \dots [f_{2k-1}, f_{2k}] \in I_r$ for all $f_i \in F\mathcal{G}_r$. Indeed, we may assume without loss of generality that $f_i \in \mathcal{G}_r$ for all i . Then

$$[f_1, f_2] \dots [f_{2k-1}, f_{2k}] = f_1 f_2 \dots f_{2k} ((f_1, f_2) + 1) \dots ((f_{2k-1}, f_{2k}) + 1).$$

It is clear that, for each s , $(f_{2s-1}, f_{2s}) = \prod_t c_{i_{st} j_{st}}$ for some commutators $c_{i_{st} j_{st}} = (y_{i_{st}}, y_{j_{st}})$. Let $d_{i_{st} j_{st}} = c_{i_{st} j_{st}} + 1$; then $c_{i_{st} j_{st}} = d_{i_{st} j_{st}} - 1$. We have

$$(f_{2s-1}, f_{2s}) + 1 = \prod_t c_{i_{st} j_{st}} + 1 = \left(\prod_t (d_{i_{st} j_{st}} - 1) \right) + 1 = \prod_t d_{i_{st} j_{st}} + \dots + \sum_{t < t'} d_{i_{st} j_{st}} d_{i_{st'} j_{st'}} + \sum_t d_{i_{st} j_{st}}.$$

It follows that the product $((f_1, f_2) + 1) \dots ((f_{2k-1}, f_{2k}) + 1)$ can be written as a sum of products of the form

$$(7) \quad d_{q_1 q_2} \dots d_{q_{2\ell-1} q_{2\ell}} = ((y_{q_1}, y_{q_2}) + 1) \dots ((y_{q_{2\ell-1}}, y_{q_{2\ell}}) + 1)$$

where $\ell \geq k$. Since $2\ell \geq 2k > r$, in the product (7) we have $q_t = q_{t'}$ for some $t < t'$. It follows that each product (7) belongs to I_r and so does the product $((f_1, f_2) + 1) \dots ((f_{2k-1}, f_{2k}) + 1)$. Hence, $[f_1, f_2] \dots [f_{2k-1}, f_{2k}] \in I_r$, as claimed.

For any $u \in F\mathcal{G}$, let $\bar{u} = u + I \in F\mathcal{G}/I$. Since one can view the algebra $F\mathcal{G}_r/I_r$ as a subalgebra of $F\mathcal{G}/I$, we also write $\bar{u} = u + I_r \in F\mathcal{G}_r/I_r$ for $u \in F\mathcal{G}_r$.

By the claim above, $[\bar{f}_1, \bar{f}_2] \dots [\bar{f}_{2k-1}, \bar{f}_{2k}] = 0$ for all $\bar{f}_i \in H$. Hence, by Corollary 2.2, we have $[u_1, \dots, u_{2k+1}] = 0$ for all $u_i \in G \otimes H$, that is,

$$[u_1, \dots, u_{m+n-1}] = 0$$

for all $u_1, \dots, u_{m+n-1} \in A$, as required.

Further, take $v_1 = \bar{y}_1 \otimes 1$, $v_i = \bar{y}_i \otimes \bar{y}_{i-1}$ ($i = 2, \dots, 2m' - 1$), $v_{2m'} = \bar{y}_{2m'} \otimes 1$, $w_1 = \bar{y}_{2m'+1} \otimes 1$, $w_j = \bar{y}_{2m'+j} \otimes \bar{y}_{2m'+j-3}$ ($j = 2, \dots, 2n' - 1$), $w_{2n'} = \bar{y}_{2m'+2n'} \otimes 1$. Note that $[\bar{y}_i, \bar{y}_j] = \bar{y}_i \bar{y}_j ((\bar{y}_j, \bar{y}_i) + 1) = \bar{y}_i \bar{y}_j ((\bar{y}_i, \bar{y}_j) + 1)$. By Corollary 2.3, we have

$$\begin{aligned} [v_1, \dots, v_{2m'}][w_1, \dots, w_{2n'}] &= [\bar{y}_1, \bar{y}_2] \dots [\bar{y}_{2m'-1}, \bar{y}_{2m'}][\bar{y}_{2m'+1}, \bar{y}_{2m'+2}] \dots [\bar{y}_{2m'+2n'-1}, \bar{y}_{2m'+2n'}] \\ &\quad \otimes [\bar{y}_1, \bar{y}_2] \dots [\bar{y}_{2m'-3}, \bar{y}_{2m'-2}][\bar{y}_{2m'-1}, \bar{y}_{2m'}] \dots [\bar{y}_{2m'+2n'-5}, \bar{y}_{2m'+2n'-4}] \\ &= \bar{y}_1 \bar{y}_2 \dots \bar{y}_{2m'+2n'} ((\bar{y}_1, \bar{y}_2) + 1) \dots ((\bar{y}_{2m'+2n'-1}, \bar{y}_{2m'+2n'}) + 1) \\ &\quad \otimes \bar{y}_1 \bar{y}_2 \dots \bar{y}_{2m'+2n'-4} ((\bar{y}_1, \bar{y}_2) + 1) \dots ((\bar{y}_{2m'+2n'-5}, \bar{y}_{2m'+2n'-4}) + 1) \end{aligned}$$

so, by Lemma 2.6, $[v_1, \dots, v_{2m'}][w_1, \dots, w_{2n'}] \neq 0$, as required.

This completes the proof of Theorem 1.4. \square

Proof of Theorem 1.3. Let A be the algebra described in Theorem 1.4. Define a homomorphism $\phi : F\langle X \rangle \rightarrow A$ by

$$\phi(x_i) = \begin{cases} v_i & \text{if } i = 1, \dots, m; \\ w_{i-m} & \text{if } i = m+1, \dots, m+n; \\ 0 & \text{if } i > m+n. \end{cases}$$

Then, on one hand, $\phi(T^{(m+n-1)}) = 0$ by the item i) of Theorem 1.4. On the other hand,

$$\phi([x_1, \dots, x_m][x_{m+1}, \dots, x_{m+n}]) = [v_1, \dots, v_m][w_1, \dots, w_n] \neq 0$$

by the item ii) of Theorem 1.4 so $\phi(T^{(m)}T^{(n)}) \neq 0$. It follows that

$$T^{(m)}T^{(n)} \not\subseteq T^{(m+n-1)},$$

as required. \square

Remarks. 1. For each $\ell \geq 1$, one can choose elements $z_1, \dots, z_{2\ell}$ in the algebra A described in Theorem 1.4 in such a way that

$$[v_1, \dots, v_m][w_1, \dots, w_n][z_1, z_2] \dots [z_{2\ell-1}, z_{2\ell}] \neq 0$$

in A . For instance, if $\text{char } F \neq 2$ then one can choose $z_i = e_{m+n+i} \otimes 1$ ($i = 1, \dots, 2\ell$). It follows that if $m = 2m'$ and $n = 2n'$ are even positive integers then, for each $\ell \geq 1$,

$$T^{(m)}T^{(n)}(T^{(2)})^\ell \not\subseteq T^{(m+n-1)}.$$

2. Let $X_k = \{x_1, x_2, \dots, x_k\}$ and let $F\langle X_k \rangle$ be the free unital associative F -algebra freely generated by X_k . Let $T_k^{(n)} = T^{(n)}(F\langle X_k \rangle)$ be the two-sided ideal of $F\langle X_k \rangle$ generated by all commutators $[a_1, a_2, \dots, a_n]$ ($a_i \in F\langle X_k \rangle$). If $k \geq m+n$ then Theorem 1.3 holds for the ideals $T_k^{(n)}$, with the same proof. However, Theorem 1.3 fails, in general, for small k : for instance, one can check that if $k \leq 3$ then $T_k^{(2)}T_k^{(2)} \subset T_k^{(3)}$. Moreover, Dangovski [6, Theorem 3.1] has recently proved that $T_2^{(m)}T_2^{(n)} \subset T_2^{(m+n-1)}$ for all $m, n \geq 2$ so Theorem 1.3 always fails for $k = 2$.

3. To prove Theorem 1.4 one can choose the algebra A different from one used in our proof. For example, let F be any field and let $r = m+n-4 = 2(m'+n'-2)$. Let $A = F\langle X \rangle / T^{(3)} \otimes F\langle X_r \rangle / T_r^{(3)}$ where $X_r = \{x_1, \dots, x_r\}$ and $T_r^{(3)} = T^{(3)}(F\langle X_r \rangle) = T^{(3)} \cap F\langle X_r \rangle$. Then A satisfies the conditions i) and ii) of Theorem 1.4; one can check this using a description of a basis of $F\langle X \rangle / T^{(3)}$ over F . Such a description can be deduced, for instance, from [3, Proposition 3.2] or found (if $\text{char } F \neq 2$) in [4, Proposition 9].

Our choice of the algebra A in the proof of Theorem 1.4 was made with a purpose to have the paper self-contained.

4. The tensor products of the form $E \otimes E_r \otimes \cdots \otimes E_s$ were used to study the polynomial identities of Lie nilpotent associative algebras over a field of characteristic 0 by Drensky [8, Section 5].

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